The distribution of registration error of a fiducial marker in rigid-body point-based registration

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ABSTRACT

Many image-guidance surgical systems rely on rigid-body, point-based registration of fiducial markers attached to the patient. Marker locations in image space and physical space are used to provide the transformation that maps a point from one space to the other. Target registration error (TRE) is known to depend on the fiducial localization error (FLE), and the fiducial registration error (FRE) of a set of markers, though a poor predictor of TRE, is a useful predictor of FLE. All fiducials are typically weighted equally for registration purposes, but it is also a common practice to ignore a marker at position \( r \) by zeroing its weight when its individual error, \( \text{FRE}(r) \), is high in an effort to reduce TRE. The idea is that such markers are likely to have been compromised, i.e., perturbed badly between imaging and surgery. While ignoring a compromised marker may indeed reduce TRE, the expected effect of ignoring an uncompromised marker is to increase TRE. There is unfortunately no established method for deciding whether a given marker is likely to have been compromised. In order to make this decision, it is necessary to know the probability distribution \( p(\text{FRE}(r)) \), which has not been heretofore determined. With such a distribution, it may be possible to identify a compromised marker and to adjust its weight in order to improve the expected TRE. In this paper we derive an approximate formula for \( p(\text{FRE}(r)) \) accurate to first order in FLE. We show by means of numerical simulations that the approximation is valid.

Keywords: Fiducial registration error, accuracy, error distribution, point-based, registration.

1. INTRODUCTION

Fiducial markers used for image-guided surgery provide the transformation between image and physical space. Depending on the accuracy desired for the surgery, the fiducial markers may be bone-implanted, skin-affixed, or fitted to the patient using a rigid frame. The fiducial markers are chosen such that they can be localized in both image and physical spaces. Rigid-body point-based registration is performed using the positions of the fiducial markers in image and in physical space to provide a transformation that maps any point from one space to the other [1]. The transformation is unique and provides perfect registration if the true locations of the corresponding fiducials are provided. In real world, however, there is usually an error in localizing the fiducials, known as fiducial localization error (FLE) that results in an imperfect registration, and, as a result, registration errors—fiducial registration error (FRE) and target registration error (TRE). FRE is defined as the error in aligning corresponding fiducials after the registration, and TRE is defined as the error in aligning corresponding targets after the registration.

TRE is an important measure in defining the accuracy of a fiducial system during the surgery. Estimates are made based on validation of the fiducial system using skulls or other target systems [2, 3]. Approximations to first order in FLE have been published for the expected squared value and the distribution of TRE [4, 5]. It is difficult to measure the actual TRE value at a target location during a surgery, but it is a simple matter to measure individual and overall FRE during the surgery every time the fiducials are localized and registered. The overall FRE provides an estimate of how well the overall fiducial localization has been done. It is not possible though to determine whether a certain fiducial has been accurately localized or whether it has been compromised (moved, bent, etc.), either of which could possibly affect the TRE. It has been a common practice to remove from the registration process a fiducial marker at position \( r \) whose individual error, \( \text{FRE}(r) \), is high [6-8]. That marker is ignored, while the remaining markers are used for the registration. This approach is based on the assumption that a fiducial marker with high FRE is possibly compromised, and that removing it will reduce TRE. This approach is reasonable when FRE is extremely high, but “extremely high” has never been quantified. If on the other hand a marker with high FRE has not been compromised, then ignoring that fiducial marker during the registration is unwarranted and will tend to increase TRE. Thus it becomes desirable to find a
2. THE MODEL

Suppose there are \( N \) fiducial points. Let \( X \) and \( Y \) be \( N \)-by-\( K \) matrices representing the positions of \( N \) corresponding fiducial markers in two \( K \)-dimensional spaces that need to be registered. Corresponding rows of \( X \) and \( Y \) contain the vector positions of two corresponding fiducial points, represented as \( \{x'_i\} \) and \( \{y'_i\} \) respectively, \( i = 1, 2, ..., N \). (Note on notation in this work: We use a nonbold font for matrices and scalars, and a bold font for vectors. All vectors are column vectors unless they are transposed. Components of matrices and vectors are in nonbold font as they are scalars. Angle brackets \( \langle \cdot \rangle \) are used to represent expected values.) In this work we derive formulas for \( K = 3 \), which applies to the typical application, in which physical fiducials are located in three-dimensional space, but the two-dimensional case can be easily handled as well by our method. Sibson found in his work in 1979 \([9]\) that to first order in FLE the distribution of registration errors in a system is independent of the gross rigid-body motion, \( i.e. \), rotation and translation, and depends only on the configuration of fiducials \( X \) in one space, and the localization errors. This fact lets us assume that there is no true rigid-body motion between \( X \) and \( Y \), but that they differ only by the perturbations caused by a FLE, drawn from an error population with zero mean. He assumed a normal distribution of errors and also simplified things by setting the localization error in one space to zero. In 1985 \([10]\) Langron and Collins showed that if there are localization errors in both spaces with expected FLE in the \( X \) and \( Y \) spaces equal to \( \{\text{FLE}_X\} \) and \( \{\text{FLE}_Y\} \) respectively, then if we set the localization error in \( X \) to zero, then the error in the other space, \( Y \), should be set to \( \{\text{FLE}_X\} = \{\text{FLE}_X^2\} \). For the purpose of error analysis we can therefore perturb \( X \) using the Langron and Collins approach in order to get the perturbed set of fiducial points \( Y \). Following Sibson’s formalism we have

\[
Y = X + \epsilon F ,
\]

where \( \epsilon \) is a positive dimensionless constant that is small enough to allow us to ignore higher order terms in the derivations that follow, and \( F \) is the \( N \)-by-\( K \) perturbation matrix representing the total FLE. The elements of \( F \) are independent, identically distributed random variables that are drawn from a zero-mean normal distribution \( \mathcal{N}(0, \sigma) \) where \( \sigma^2 = \langle \text{FLE}^2 \rangle / K \). The choice of origin for \( X \) is arbitrary as it is just a translation effect in the overall transformation. So Sibson assumes \( X \) to be centered at origin. Thus

\[
\sum_{a=1}^{K} X_{i,a} = 0 .
\]

We can also choose the coordinate system to be the principal axes of the configuration of fiducial points without affecting the system. This reorientation is accomplished by finding the singular-value decomposition (SVD) for \( X \), such that \( X = U \Lambda V^T \), and then setting \( X = U \Lambda \). (Note: The reorientation of the coordinate system and the choice of the center of \( X \) as our origin are done merely to simplify the derivation that follows. They can easily be undone at the end.)

The problem of rigid-body point-based registration of \( X \) and \( Y \) is that of finding an orthogonal rotation matrix \( R \) and a translation vector \( t \) such that they together minimize the sum of individual FLE’s, given by

\[
G = \sum_{i=1}^{N} \left( (R x_i + t) - y_i \right)^T .
\]

With our assumption of zero true motion, the translation vector, \( t \), is simply the mean displacement between the centers of the two sets of fiducials caused by the localization error. For our set of fiducials,

\[
t' = \epsilon \bar{\gamma} F / N .
\]

The rotation matrix, \( R \), as given by Schönemann \([11]\) that minimizes \( G \) is \( R = BA \), where \( \hat{X} \) and \( \hat{Y} \) are demeaned \( X \) and \( Y \) respectively and \( ADB^T \) is the SVD of \( \hat{Y}' \hat{X} \). We can express \( R \) as a power series in \( \epsilon \) as follows

\[
R = R^{(0)} + \epsilon R^{(1)} + O(\epsilon^2) .
\]

When \( \epsilon = 0 \), \( X = Y \). Therefore \( R \) is the identity matrix \( I \), \( i.e. \), \( R^{(0)} = I \). By using the orthogonal property of \( R \), we see that \( R^{(1)} \) is antisymmetric, \( i.e. \), \( R^{(1)} = -R^{(0)} \). Ignoring higher order terms, we can see that
If we define $Q = U' \hat{F}$, where $\hat{F}$ is demeaned $F$, then from the work of Goodall [12] we see that

$$R^{(i)} = \frac{\Lambda_2 Q_2 - \Lambda_3 Q_3}{\Lambda_2^2 + \Lambda_3^2}.$$  \hspace{1cm} (7)

3. DERIVATION OF THE FRE DISTRIBUTION

We begin our derivation of the FRE distribution by analyzing the vector fiducial registration error, $\text{FRE}(r)$, for a fiducial marker $r$. It is given by

$$\text{FRE}(r) = Rr + t - (r + \varepsilon f) = (R - I)r + t - \varepsilon f,$$  \hspace{1cm} (8)

where $f$ represents the fiducial localization error vector at $r$. Expanding up to first order and ignoring higher order terms, and using (6),

$$\text{FRE}(r) = R^{(i)}r + t - f = \Omega \times r + t - f,$$  \hspace{1cm} (9)

where we follow the same definition as in [5] for $\Omega$, which is for a three dimensional space ($K = 3$)

$$\Omega = \begin{bmatrix} R^{(1)}_{11} & R^{(1)}_{12} & R^{(1)}_{13} \\ R^{(1)}_{21} & R^{(1)}_{22} & R^{(1)}_{23} \\ R^{(1)}_{31} & R^{(1)}_{32} & R^{(1)}_{33} \end{bmatrix}.$$  \hspace{1cm} (10)

To simplify the following derivations, we enforce the following restriction on the values of the subscripts $j, k, l$:

$$\{j,k,l\} = \{1,2,3\}, \{2,3,1\}, \text{or} \{3,1,2\}.$$  \hspace{1cm} (11)

With this restriction, the definition of $\Omega$ becomes $\Omega_{ij} = R^{(i)}_{ij}$.

3.1 Resolution into independent components

Let us resolve $\text{FRE}(r)$ into three orthogonal components—$\text{FRE}_r, \text{FRE}_v, \text{FRE}_w$—along three directions, $\hat{r}, \hat{v},$ and $\hat{w}$. We will choose $\hat{r}$ to be a unit vector along the radial direction, $\hat{v}$ to be a unit vector chosen perpendicular to $\hat{r}$, and $\hat{w}$ to be a unit vector perpendicular to both $\hat{r}$ and $\hat{v}$. These vectors are chosen such that the components of $\text{FRE}(r)$ in these directions are uncorrelated as described later. If $r$ is the magnitude of the vector $r$, then the three components are given by the following three equations:

$$\text{FRE}_r(r) = (\Omega \times r + t - f) \cdot \hat{r} = t \cdot \hat{r} - f \cdot \hat{r},$$  \hspace{1cm} (12)

$$\text{FRE}_v(r) = (\Omega \times r + t - f) \cdot \hat{v} = r \Omega \cdot \hat{w} + t \cdot \hat{v} - f \cdot \hat{v},$$  \hspace{1cm} (13)

$$\text{FRE}_w(r) = (\Omega \times r + t - f) \cdot \hat{w} = -r \Omega \cdot \hat{v} + t \cdot \hat{w} - f \cdot \hat{w}. $$  \hspace{1cm} (14)

The individual FRE at the fiducial position $r$ is given by

$$\text{FRE}(r) = \text{FRE}_r \hat{r} + \text{FRE}_v \hat{v} + \text{FRE}_w \hat{w}. $$  \hspace{1cm} (15)

We wish to find a direction for $\hat{v}$ and $\hat{w}$ such that $\hat{r} \cdot \hat{v} = 0, \hat{r} \cdot \hat{w} = 0,$ and $\hat{r} \times \hat{v} = \hat{w}$, and the individual components of FRE are uncorrelated, i.e.,

$$\langle \text{FRE}_r(r) \text{FRE}_v(r) \rangle = \langle \text{FRE}_r(r) \text{FRE}_w(r) \rangle = \langle \text{FRE}_v(r) \text{FRE}_w(r) \rangle = 0.$$  \hspace{1cm} (16)

3.2 Cross correlation terms

We begin by analyzing the cross correlation among the components of translation and rotation.

$$\langle t_i t_j \rangle = \frac{1}{N^2} \left( \sum_{w=1}^{N} F_w \right) \left( \sum_{y=1}^{N} F_y \right) = \frac{\sigma^2}{N} \delta_{ij}.$$  \hspace{1cm} (17)
where $\delta$ is the Kronecker delta function defined as $\delta_{ij} = 1$ and $\delta_{ij} = 0$ for $i \neq j$. This is based on the assumption that the elements of $F$ are independent and are drawn from a zero-mean normal distribution with variance $\sigma^2$, and so $\langle F_i F_j \rangle = \delta_{ij} \sigma^2$.

It was shown in [5] that $\langle t_i \Omega_j \rangle = 0$ and $\langle \Omega_i \Omega_j \rangle = \omega_j^2 \delta_{ij}$, where, using the restriction on $j,k,l$ given above, we define

$$\omega_j^2 = \frac{\sigma^2}{\Lambda_{kk}^i + \Lambda_{ll}^j}.$$  

Looking at $\langle F_i \Omega_j \rangle$ where $F_i$ is the fiducial localization error for the fiducial $r$ under consideration along the axis $i$,

$$\langle F_i \Omega_j \rangle = \left\langle F_i \left( \Lambda_{kk}^i \Omega_{ij} - \Lambda_{ll}^j \Omega_{ij} \right) \right\rangle = \left\langle \left( \sum_{k=1}^{K} (\Lambda_{kk}^i U_{ij} F_i \hat{F}_j - \Lambda_{ll}^j U_{ij} F_i \hat{F}_j) \right) \right\rangle \left( \Lambda_{kk}^i + \Lambda_{ll}^j \right).$$

Using the definitions of $F_i$ and $\hat{F}_j$, we see that

$$\langle F_i \hat{F}_j \rangle = \left\langle F_i \left( 1 - \frac{1}{N} \sum_{k=1}^{K} F_k \right) \right\rangle = \sigma^2 \left( \delta_{ab} \delta_{ij} - \frac{1}{N} \sum_{a=1}^{N} \sigma_{ab}^2 \delta_{ij} \right) = \sigma^2 \delta_{ab} \left( \delta_{ij} - \frac{1}{N} \right).$$

Applying this to (17) and simplifying we have

$$\langle F_i \Omega_j \rangle = \frac{\sigma^2}{\Lambda_{kk}^i + \Lambda_{ll}^j} \sum_{k=1}^{K} \Lambda_{kk}^i U_{ij} \delta_{ij} \left( \delta_{ab} - \frac{1}{N} \right) = \sigma^2 \left( \frac{\delta_{ab} \delta_{ij} - \frac{1}{N} \sum \delta_{ab} \delta_{ij}}{\Lambda_{kk}^i + \Lambda_{ll}^j} \right).$$

Looking at $\langle F_i t_j \rangle$, we find that

$$\langle F_i t_j \rangle = \left\langle F_i \left( \sum_{k=1}^{K} F_k \right) \right\rangle = \sigma^2 \left( \sum_{a=1}^{N} \sum \delta_{ab} \delta_{ij} \right) = \frac{\sigma^2}{N} \delta_{ij}.$$

### 3.3 Correlation of FRE components and choice of direction for $\hat{v}$ and $\hat{w}$

Using the results of cross correlation terms, we have

$$\langle \text{FRE}_r (r) \rangle = \langle (t \cdot \hat{r} - f_\Omega \cdot \hat{r}) (r \Omega \cdot \hat{v} + t \cdot \hat{v} - f \cdot \hat{v}) \rangle$$

$$= \frac{\sigma^2}{N} \sum_{a=1}^{N} \sum \delta_{ab} \delta_{ij} r_{ij} \hat{w}_{ij} - \sigma^2 \sum_{a=1}^{N} \sum \delta_{ab} \delta_{ij} r_{ij} \hat{w}_{ij} - r \sigma^2 \sum_{a=1}^{N} \sum \delta_{ab} \delta_{ij} \hat{w}_{ij} + \sigma^2 \sum_{a=1}^{N} \sum \delta_{ab} \delta_{ij} \hat{w}_{ij}$$

$$= -r \sigma^2 \sum_{a=1}^{N} \sum \left( \frac{\delta_{ab} \delta_{ij} - \frac{1}{N} \sum \delta_{ab} \delta_{ij}}{\Lambda_{kk}^i + \Lambda_{ll}^j} \right) \hat{w}_{ij}.$$ 

For three dimensional situation, i.e., $K = 3$,

$$\sum_{a=1}^{N} \sum \left( \frac{\delta_{ab} \delta_{ij} - \frac{1}{N} \sum \delta_{ab} \delta_{ij}}{\Lambda_{kk}^i + \Lambda_{ll}^j} \right) \hat{w}_{ij} = \frac{r \hat{F}_i \hat{w}_2 - \frac{1}{N} \sum \delta_{ab} \delta_{ij} \hat{w}_j}{\Lambda_{kk}^i + \Lambda_{ll}^j}$$

$$= r \left( \frac{\hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} - \frac{r \hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} - \frac{r \hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} + \frac{r \hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} + \frac{r \hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} - \frac{r \hat{F}_i \hat{w}_2}{\Lambda_{kk}^i + \Lambda_{ll}^j} \right)$$

$$= 0.$$
This makes \( \langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle = 0 \). We can similarly prove that \( \langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle = 0 \). This is valid for any \( \mathbf{v} \) and \( \mathbf{w} \) that are perpendicular to \( \mathbf{r} \). We will begin analyzing \( \langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle \) now, which will impose a constraint on the choice of \( \mathbf{v} \) and \( \mathbf{w} \) for the terms to become zero.

\[
\langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle = \langle (r \mathbf{\Omega} \cdot \mathbf{r} + t \cdot \mathbf{\hat{v}} - f \cdot \mathbf{\hat{w}}) (r \mathbf{\Omega} \cdot \mathbf{r} + t \cdot \mathbf{\hat{v}} - f \cdot \mathbf{\hat{w}}) \rangle
\]

\[
= -r^2 \sum_{i=1}^{K} \sum_{j=1}^{K} \langle \mathbf{t}_i \rangle \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j + r \sum_{i=1}^{K} \sum_{j=1}^{K} \langle \mathbf{F}_i \rangle \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j - r \sum_{i=1}^{K} \sum_{j=1}^{K} \langle \mathbf{t}_i \rangle \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j + r \sum_{i=1}^{K} \sum_{j=1}^{K} \langle \mathbf{F}_i \rangle \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j + \sum_{i=1}^{K} \sum_{j=1}^{K} \langle \mathbf{t}_j \rangle \mathbf{\hat{v}}_j \mathbf{\hat{w}}_j \tag{23}
\]

Using the results derived before for cross correlation terms, we see that

\[
\langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle = \sigma^2 \sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\delta_{ij} \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j}{\Lambda_{ii} + \Lambda_{jj}} = r^2 \sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\delta_{ij} \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j}{\Lambda_{ii} + \Lambda_{jj}} \tag{24}
\]

Again considering the three dimensional case, we will expand each of the three terms on the right-hand side of the above equation as follows:

\[
\sum_{i=1}^{K} \sum_{j=1}^{K} \frac{\delta_{ij} \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j}{\Lambda_{ii} + \Lambda_{jj}} = \sigma^2 \left( \frac{\hat{\mathbf{v}}_1 \mathbf{\hat{v}}_1}{\Lambda_{11} + \Lambda_{11}} + \frac{\hat{\mathbf{v}}_2 \mathbf{\hat{v}}_2}{\Lambda_{22} + \Lambda_{22}} + \frac{\hat{\mathbf{v}}_3 \mathbf{\hat{v}}_3}{\Lambda_{33} + \Lambda_{33}} \right). \tag{25}
\]

\[
\sum_{i=1}^{K} \sum_{j=1}^{K} \left( \frac{\delta_{ij} r_i - \delta_{ij} r_j}{\Lambda_{ii} + \Lambda_{jj}} \right) \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j = r \left( \frac{\hat{\mathbf{v}}_1 \mathbf{\hat{w}}_1}{\Lambda_{11} + \Lambda_{11}} - \frac{\hat{\mathbf{v}}_2 \mathbf{\hat{w}}_2}{\Lambda_{22} + \Lambda_{22}} + \frac{\hat{\mathbf{v}}_3 \mathbf{\hat{w}}_3}{\Lambda_{33} + \Lambda_{33}} \right). \tag{26}
\]

\[
\sum_{i=1}^{K} \sum_{j=1}^{K} \left( \frac{\delta_{ij} r_i - \delta_{ij} r_j}{\Lambda_{ii} + \Lambda_{jj}} \right) \mathbf{\hat{v}}_i \mathbf{\hat{w}}_j = r \left( \frac{\hat{\mathbf{v}}_1 \mathbf{\hat{w}}_1}{\Lambda_{11} + \Lambda_{11}} + \frac{\hat{\mathbf{v}}_2 \mathbf{\hat{w}}_2}{\Lambda_{22} + \Lambda_{22}} + \frac{\hat{\mathbf{v}}_3 \mathbf{\hat{w}}_3}{\Lambda_{33} + \Lambda_{33}} \right). \tag{27}
\]

Applying these in (24) gives us,

\[
\langle \text{FRE}_u (\mathbf{r}) \rangle \text{FRE}_u (\mathbf{r}) \rangle = r^2 \sigma^2 \left( \frac{\hat{\mathbf{v}}_1 \mathbf{\hat{v}}_1}{\Lambda_{22} + \Lambda_{22}} + \frac{\hat{\mathbf{v}}_2 \mathbf{\hat{v}}_2}{\Lambda_{33} + \Lambda_{33}} + \frac{\hat{\mathbf{v}}_3 \mathbf{\hat{v}}_3}{\Lambda_{11} + \Lambda_{11}} \right). \tag{28}
\]

This term becomes zero for a certain \( \mathbf{\hat{v}} \) and \( \mathbf{\hat{w}} \). The problem can be converted to solving the quadratic equation

\[
r_j v_j v_j (\alpha_j^2 - \alpha_j^2) + r_j v_j v_j (\alpha_j^2 - \alpha_j^2) + r_j v_j v_j (\alpha_j^2 - \alpha_j^2) = 0. \tag{29}
\]

Fitzpatrick et al. [5] found a solution for \( \mathbf{\hat{v}} \) and \( \mathbf{\hat{w}} \) that makes this same term zero in order to make \( \langle \text{TRE}, \text{TRE}_u \rangle \) equal zero. With the restriction on the values of \( j, k, \) and \( l \) given above, the solution can be stated as follows:

Case 1: At least one of the components of \( \mathbf{r} \) is zero. Suppose \( r_j = 0 \), then there are two possible solutions, namely:

i. Set \( v_j = 0 \), choose any solution for \( v_k \) and \( v_l \) that satisfies \( \mathbf{r} \cdot \mathbf{v} = 0 \), and then normalize the vector \( \mathbf{v} \).

ii. Set \( v_j = 1 \) and \( v_k = v_l = 0 \).
We can choose one of these two solutions for \( \hat{v} \) and the other for \( \hat{w} \).

Case 2: If \( \omega_j^2 = \omega_k^2 \), then there are two simple solutions as follows:

i. Set \( v_j = 0 \), choose any solution for \( v_j \) and \( v_k \) that satisfies \( r \cdot v = 0 \), and then normalize the vector \( v \).

ii. Set \( v_j = v_k = 0 \).

Like for case 1 we can choose one of these solutions for \( \hat{v} \) and the other for \( \hat{w} \).

Case 3: This is the general case when none of the components of \( r \) is zero and all the \( \omega_j^2 \) are distinct. We assume \( v_j = 1 \) for both the possible solutions. The two possible solutions for \( v_j \) are the two solutions to the following quadratic equation:

\[
r r_j r_k \left( \omega_j^2 - \omega_k^2 \right) v_j^2 + \left[ \left( r_j^2 + r_k^2 \right) \left( \omega_j^2 - \omega_k^2 \right) + \left( r_j^2 + r_k^2 \right) \left( \omega_k^2 - \omega_j^2 \right) \right] v_j + r r_j r_k \left( \omega_k^2 - \omega_j^2 \right) = 0.
\]  

(30)

The two solutions for \( v_j \) are obtained by substituting each possible value of \( v_j \) and \( v_k \) in \( r \cdot v = 0 \) and solving for \( v_j \). One of the two solutions is then chosen for obtaining \( \hat{v} \) and the other for \( \hat{w} \).

3.4 Expected values

The expected value of the square of the magnitude of \( \text{FRE} \) at \( r \), \( \langle \text{FRE}^2 (r) \rangle \) is given by

\[
\langle \text{FRE}^2 (r) \rangle = \langle \text{FRE}_r^2 (r) \rangle + \langle \text{FRE}_w^2 (r) \rangle + \langle \text{FRE}_h^2 (r) \rangle.
\]  

(31)

Since we are ignoring higher order terms, each component of \( \text{FRE}(r) \) is a linear combination of elements of \( F \), which are normally distributed with zero mean. This implies that the components of \( \text{FRE}(r) \) also have zero mean. So the respective variances of its components—\( \sigma_r^2 \), \( \sigma_v^2 \), and \( \sigma_w^2 \)—are same as their mean squared values. We can find these values from (10), (11), and (12) as follows:

\[
\sigma_r^2 = \frac{3}{N} \sum_{i=1}^{k} \sum_{j=1}^{k} \delta_j r_i \hat{r}_j^2 + \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \delta_j r_i \hat{r}_j = \sigma^2 \left( \frac{N-1}{N} \right).
\]  

(32)

Suppose we define

\[
\omega_j^2 = \sum_{i=1}^{k} \omega_j^2 \hat{r}_i^2, \quad \omega_v^2 = \sum_{i=1}^{k} \omega_v^2 \hat{v}_i^2, \quad \text{and} \quad \omega_w^2 = \sum_{i=1}^{k} \omega_w^2 \hat{w}_i^2.
\]  

(33)

Then

\[
\sigma_r^2 = \langle \text{FRE}_r^2 (r) \rangle = \langle \left( (r \cdot \hat{r} + t \cdot \hat{v}) - (r \cdot \hat{r} + t \cdot \hat{v}) \right)^2 \rangle
\]

\[
= r^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \Omega_i \Omega_j \right) \hat{w}_i \hat{w}_j + \sum_{i=1}^{k} \sum_{j=1}^{k} \left( t t_j \right) \hat{v}_i \hat{v}_j + \sum_{i=1}^{k} \sum_{j=1}^{k} \left( F_i F_j \right) \hat{v}_i \hat{v}_j
\]

\[
+2r \sum_{i=1}^{k} \sum_{j=1}^{k} \left( t \Omega_i \right) \hat{v}_i \hat{w}_j - 2 \sum_{i=1}^{k} \sum_{j=1}^{k} \left( t F_i \right) \hat{v}_i \hat{v}_j - 2r \sum_{i=1}^{k} \sum_{j=1}^{k} \left( F_i \Omega_j \right) \hat{v}_i \hat{w}_j
\]

\[
= r^2 \sigma_r^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \Lambda_i \Lambda_j + \Lambda_r^2 + \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \delta_j \hat{v}_i \hat{v}_j + \sigma^2 \sum_{i=1}^{k} \sum_{j=1}^{k} \delta_j \hat{v}_i \hat{w}_j
\]

\[
= \frac{2 \sigma^2}{N} \sum_{i=1}^{k} \sum_{j=1}^{k} \delta_j \hat{v}_i \hat{v}_j - 2 \sigma^2 r \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \delta_j \hat{v}_i - \delta_j \hat{w}_i \right) \hat{w}_j
\]

\[
= r^2 \sigma_v^2 + \sigma^2 \left( \frac{N-1}{N} \right) - 2 \sigma^2 r \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \frac{\delta_j \hat{v}_i - \delta_j \hat{w}_i}{\Lambda_r^2 + \Lambda_i^2} \right) \hat{w}_j
\]

\[
= \sigma^2 \left( \frac{N-1}{N} \right) - r^2 \omega_v^2.
\]  

(34)
Now from these variances we can compute \( \langle \text{FRE}^2 (\mathbf{r}) \rangle \) to be
\[
\langle \text{FRE}^2 (\mathbf{r}) \rangle = \sigma_e^2 + \sigma_r^2 + \sigma_w^2 = 3\sigma^2 \left( \frac{N-1}{N} \right) - r^2 \left( \omega_e^2 + \omega_r^2 + \omega_w^2 \right). \tag{36}
\]
Since \( \hat{\mathbf{r}} \), \( \hat{\mathbf{v}} \), and \( \hat{\mathbf{w}} \) are orthogonal set of axes, we can see from (33) that \( \omega_e^2 + \omega_r^2 + \omega_w^2 = \omega_e^2 + \omega_r^2 + \omega_w^2 \). This implies
\[
\omega_e^2 + \omega_r^2 = \omega_e^2 + \omega_r^2 + \omega_w^2 - \frac{r_1^2 \omega_e^2 + r_2^2 \omega_r^2 + r_3^2 \omega_w^2}{r_1^2 + r_2^2 + r_3^2} = \omega_e^2 \left( r_1^2 + r_2^2 \right) + \omega_r^2 \left( r_1^2 + r_2^2 \right) + \omega_w^2 \left( r_1^2 + r_2^2 \right). \tag{37}
\]
Using this result in (36) yields,
\[
\langle \text{FRE}^2 (\mathbf{r}) \rangle = 3\sigma^2 \left( \frac{N-1}{N} \right) - \left( \omega_e^2 \left( r_1^2 + r_2^2 \right) + \omega_r^2 \left( r_1^2 + r_2^2 \right) + \omega_w^2 \left( r_1^2 + r_2^2 \right) \right). \tag{38}
\]
If we express the distance of \( \mathbf{r} \) from axis \( j \) as \( d_j \), then \( d_j = r_1^2 + r_j^2 \) where \( j, k, \) and \( l \) follow the restriction defined above. Then we can express \( \langle \text{FRE}^2 (\mathbf{r}) \rangle \) as
\[
\langle \text{FRE}^2 (\mathbf{r}) \rangle = 3\sigma^2 \left( \frac{N-1}{N} \right) - \sum_{j=1}^{3} d_j^2 \omega_j^2. \tag{39}
\]
\( \Lambda_{jk} + \Lambda_{kl} \) is the sum of squared distance of the fiducial points from axis \( j \). Using the definition of \( \omega_j^2 \) from (16), we can rewrite the above expression for \( \langle \text{FRE}^2 (\mathbf{r}) \rangle \) as follows:
\[
\langle \text{FRE}^2 (\mathbf{r}) \rangle = \sigma^2 \left( 3 \left( \frac{N-1}{N} \right) - \sum_{j=1}^{3} d_j^2 \right). \tag{40}
\]
Substituting \( \langle \text{FLE}^2 \rangle = 3\sigma^2 \), we get the same result found by Fitzpatrick et al. [4]. Thus, the variances we derived here agree with the previous results.

### 3.5 Expected values in any arbitrary direction

We will now derive the expected square value of the component of \( \text{FRE} \) at \( \mathbf{r} \) in any arbitrary direction \( \hat{\mathbf{a}} \). From (13) we can say that the component of \( \text{FRE}(\mathbf{r}) \) in any direction \( \hat{\mathbf{a}} \) is
\[
\text{FRE}_u (\mathbf{r}) = \text{FRE}_r \mathbf{a}_r + \text{FRE}_v \mathbf{a}_v + \text{FRE}_w \mathbf{a}_w, \quad (41)
\]

where \( \mathbf{a}_r = \mathbf{a} \cdot \mathbf{r} \), \( \mathbf{a}_v = \mathbf{a} \cdot \mathbf{v} \), and \( \mathbf{a}_w = \mathbf{a} \cdot \mathbf{w} \). This also implies that
\[
\langle \text{FRE}_r^2 (\mathbf{r}) \rangle = \langle \text{FRE}_v^2 (\mathbf{r}) \rangle + \langle \text{FRE}_w^2 (\mathbf{r}) \rangle + \langle \text{FRE}_u^2 (\mathbf{r}) \rangle.
\quad (42)
\]

Since \( \text{FRE}_r \), \( \text{FRE}_v \), and \( \text{FRE}_w \) are uncorrelated, the individual components of \( \text{FRE}_u (\mathbf{r}) \) are also uncorrelated. Thus (42) is valid. Using (32), (34), and (35) in (42) we obtain
\[
\langle \text{FRE}_u^2 (\mathbf{r}) \rangle = \sigma_r^2 \left( \frac{N-1}{N} \right) - r^2 \left( \mathbf{a}_r \mathbf{a}_u + \mathbf{a}_u \mathbf{a}_r \right). \quad (43)
\]

### 3.6 Distribution

We have shown how to decompose the vector \( \text{FRE}(\mathbf{r}) \) into three orthogonal components, namely \( \text{FRE}_r \), \( \text{FRE}_v \), and \( \text{FRE}_w \) along the \( \mathbf{r} \), \( \mathbf{v} \), and \( \mathbf{w} \) directions respectively, such that the components are mutually uncorrelated. Each component is linear in the components of \( \text{FRE} \), each of which is drawn from \( \mathcal{N}(0, \sigma) \). Thus the components are likewise normally distributed with zero means. Because they are normally distributed with zero means and are uncorrelated, they must in fact be mutually independent [13]. As a result of this independence of its components, \( \text{FRE}^2 (\mathbf{r}) \) is distributed as the sum of three chi-square variables,
\[
p \left( \text{FRE}^2 (\mathbf{r}) \right) = p \left( \sigma_r^2 X_r^2 + \sigma_v^2 X_v^2 + \sigma_w^2 X_w^2 \right), \quad (44)
\]

where \( \sigma_r^2 \), \( \sigma_v^2 \), and \( \sigma_w^2 \) are the variances of the respective components of \( \text{FRE}(\mathbf{r}) \) and are given in the previous section. Since Sibson’s work, it has been more common to report \( \text{FRE}^2 \), but, by using the general relationship for probability density functions \( p(x) = p(f(x)) df/\mathbf{d}x \) in combination with (44), the distribution for \( \text{FRE}(\mathbf{r}) \) is easily calculated:
\[
p \left( \text{FRE}(\mathbf{r}) \right) = 2 \sqrt{\sigma_r^2 X_r^2 + \sigma_v^2 X_v^2 + \sigma_w^2 X_w^2} p \left( \sigma_r^2 X_r^2 + \sigma_v^2 X_v^2 + \sigma_w^2 X_w^2 \right). \quad (45)
\]

Along an arbitrary direction \( \mathbf{a} \), \( \text{FRE}_u (\mathbf{r}) \) is the sum of three independent, zero mean, normal variables given by (41). Therefore \( \text{FRE}_u (\mathbf{r}) \) is also a zero-mean, normal variable with a variance \( \sigma_u^2 \) equal to \( \langle \text{FRE}_u^2 (\mathbf{r}) \rangle \), given by (43). Thus
\[
p \left( \text{FRE}_u (\mathbf{r}) \right) = \mathcal{N}(0, \sigma_u^2). \quad (46)
\]

### 4. SIMULATIONS

In this paper we have derived (44) and (46), which are approximate distributions of \( \text{FRE} \) of any individual fiducial \( \mathbf{r} \). We test the correctness of our results by means of simulations.

Six different values were chosen for the number of fiducials, \( N \), for the test: \( N = 3, 4, 5, 10, 15, 20 \). A fiducial matrix \( X \) with \( N \) fiducial marker locations is generated by randomly choosing \( N \) three-dimensional locations with uniform distribution inside a cube of edge 200 mm. The perturbed set of fiducial markers \( Y \) is obtained by perturbing each fiducial in \( X \) along the \( x \), \( y \), and \( z \) directions using normally distributed random variables with zero mean and variance \( \sigma^2 = 1/3 \text{mm}^2 \). This is the same model previously used by Sibson [9] and Fitzpatrick et al. [4, 5], and represent typical volume and error scales found in surgical applications. \( X \) and \( Y \) are registered using the point-based registration algorithm explained in [1], and the individual \( \text{FRE} \) values are measured.

Two sets of simulations were performed for the verification of the \( \text{FRE} \) distribution—the first to calculate our predicted distribution, as given in (44), and the second to produce a ground-truth distribution from actual registrations. For each fiducial the first simulation involved \( \mathcal{M}_1 \) iterations of generating three independent, zero-mean normal variables with variances \( \sigma_r^2 \), \( \sigma_v^2 \), and \( \sigma_w^2 \), given by (32), (34), and (35) respectively, that gives the three components of \( \text{FRE} \) at that fiducial. The corresponding values of the three components were squared and added to obtain \( \mathcal{M}_1 \) values for \( \text{FRE}^2 \) for a fiducial. The second simulation involved \( \mathcal{M}_2 \) iterations of perturbing \( X \) to get \( Y \), registering \( X \) and \( Y \), and measuring the individual \( \text{FRE}^2 \) of each fiducial. \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) were chosen to be 500,000 for our testing. The two simulations were
repeated for each of ten different sets $X$ of fiducials for each value of $N$. All simulations were performed using Matlab software (Version 7.5.0.342, R2007b, Natick, MA) on a Windows operating system.

Figure 1 shows a plot of corresponding percentile values (stars) of predicted versus ground-truth FRE values, which were obtained by taking the square root of the $\mathcal{M}_1$ and $\mathcal{M}_2$ FRE$^2$ values, for the fiducial marker with largest absolute percentage difference in the mean FRE values. The $x$ axis represents the percentile values for the FRE values obtained from the ground-truth simulation based on point-based registration algorithm in increments of 5% ranging from 5$^{th}$ to 95$^{th}$ percentile value. The $y$ axis represents the same percentiles for our predicted FRE values obtained from the simulation based on (44). The percentile plot closely follows the straight line along $x = y$ (solid line) showing that the results from the two simulations match each other closely. This strongly indicates that our predicted distribution given by (44) is valid for the configurations on which we tested it. The Kolmogorov-Smirnov test [14] was performed to compare the two distributions given by the $\mathcal{M}_1$ and $\mathcal{M}_2$ FRE$^2$ values for the fiducial markers whose percentile plots are shown in Figure 1. No significant difference was found between the two distributions for $N = 3, 4, 10, 15$ and $20$. For the case $N = 5$, there was a significant difference (K-S test, $p = 0.000336$).

Table 1 shows a comparison of the variance values for each value of $N$ along the $r$, $v$, $w$, $x$, $y$, and $z$ directions. The predicted variances along the $r$, $v$, and $w$ directions were computed using (32), (34), and (35) respectively. The predicted variance along the $x$, $y$, and $z$ directions were computed using (43) with $\hat{a} = \hat{x}$, $\hat{y}$, and $\hat{z}$ respectively. We choose these particular three vectors simply for programming convenience. We can make this choice because our fiducial configurations are random, and therefore $\hat{x}$, $\hat{y}$, and $\hat{z}$ represent arbitrary directions relative to $\hat{r}$, $\hat{v}$, and $\hat{w}$. The ground-truth variances were obtained by computing the $\mathcal{M}_2$ FRE$^2$ components along the $r$, $v$, $w$, $x$, $y$, and $z$ directions for each fiducial during the second simulation, which is based on the point-based registration algorithm. For each value of $N$, the fiducial marker that had the maximum absolute difference between the predicted and ground-truth variance values among all the 10 sets of $X$ was selected, and the variance values corresponding to that fiducial marker are reported in Table 1. We see that the predicted and ground-truth values match quite closely in all the directions.

![Figure 1](image-url)
Table 1. Variances along the \( r, v, w, x, y, \) and \( z \) directions for the fiducial marker with maximum difference (units = mm\(^2\)).

<table>
<thead>
<tr>
<th>( N )</th>
<th>Predicted Values</th>
<th>Ground-truth Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \sigma^2_r )</td>
<td>( \sigma^2_v )</td>
</tr>
<tr>
<td>3</td>
<td>0.2222</td>
<td>0.0556</td>
</tr>
<tr>
<td>4</td>
<td>0.2500</td>
<td>0.1315</td>
</tr>
<tr>
<td>5</td>
<td>0.2667</td>
<td>0.0838</td>
</tr>
<tr>
<td>10</td>
<td>0.3000</td>
<td>0.2940</td>
</tr>
<tr>
<td>15</td>
<td>0.3111</td>
<td>0.2558</td>
</tr>
<tr>
<td>20</td>
<td>0.3167</td>
<td>0.3107</td>
</tr>
</tbody>
</table>

Table 2. Mean and standard deviation of the differences in the predicted and ground-truth variance values along the \( r, v, w, x, y, \) and \( z \) directions (units = mm\(^2\)).

<table>
<thead>
<tr>
<th>( N )</th>
<th>Mean and standard deviation of the differences in the variance (predicted variance – ground-truth variance)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\sigma} ) ( \hat{\sigma} ) ( \hat{\sigma} ) ( \hat{\sigma} ) ( \hat{\sigma} )</td>
</tr>
<tr>
<td>3</td>
<td>( 0.00007 \pm 0.00046 ) ( 0.00002 \pm 0.00032 ) ( -0.00007 \pm 0.00013 ) ( 0.00004 \pm 0.00033 ) ( -0.00002 \pm 0.00019 ) ( 0.00000 \pm 0.00026 )</td>
</tr>
<tr>
<td>4</td>
<td>( -0.00005 \pm 0.00047 ) ( -0.00001 \pm 0.00022 ) ( 0.00008 \pm 0.00026 ) ( -0.00002 \pm 0.00041 ) ( 0.00001 \pm 0.00024 ) ( 0.00004 \pm 0.00035 )</td>
</tr>
<tr>
<td>5</td>
<td>( -0.00003 \pm 0.00054 ) ( 0.00001 \pm 0.00032 ) ( 0.00001 \pm 0.00031 ) ( -0.00002 \pm 0.00039 ) ( 0.00004 \pm 0.00034 ) ( -0.00002 \pm 0.00041 )</td>
</tr>
<tr>
<td>10</td>
<td>( 0.00002 \pm 0.00052 ) ( 0.00001 \pm 0.00050 ) ( 0.00000 \pm 0.00048 ) ( 0.00004 \pm 0.00052 ) ( 0.00002 \pm 0.00050 ) ( -0.00003 \pm 0.00051 )</td>
</tr>
<tr>
<td>15</td>
<td>( -0.00001 \pm 0.00057 ) ( 0.00003 \pm 0.00060 ) ( -0.00002 \pm 0.00053 ) ( 0.00007 \pm 0.00056 ) ( -0.00002 \pm 0.00060 ) ( -0.00005 \pm 0.00052 )</td>
</tr>
<tr>
<td>20</td>
<td>( -0.00006 \pm 0.00063 ) ( 0.00004 \pm 0.00057 ) ( 0.00001 \pm 0.00053 ) ( -0.00000 \pm 0.00054 ) ( -0.00001 \pm 0.00057 ) ( 0.00001 \pm 0.00055 )</td>
</tr>
</tbody>
</table>

Table 2 reports the mean and standard deviation values of the differences of the variance values along the \( r, v, w, x, y, \) and \( z \) directions corresponding to all the fiducial markers for a particular \( N \) value. For example, for the case \( N = 4 \) there are 40 fiducial markers in the 10 \( X \) matrices. There are two sets of 40 values of \( \sigma^2_r, \sigma^2_v, \sigma^2_w \), where the two sets correspond to the predicted variances and the ground-truth variances of FRE in the \( x \) direction. The column in Table 2 labeled \( \hat{\sigma} \) reports the mean and standard deviation of the differences between these two sets of values. We see that the differences are close to zero for all cases in all the directions. These small differences indicate that our approximate prediction of variance of FRE along \( r, v, w, \) and any arbitrary direction \( \hat{\sigma} \) works well for all the fiducial configurations in our test.

5. DISCUSSION

We have derived a first-order approximation for the distribution of the fiducial registration error (FRE) of any fiducial marker \( r \). We have shown that FRE can be resolved into three orthogonal components along the directions \( \hat{r}, \hat{v}, \) and \( \hat{w} \) such that the three components are independent, zero-mean normal distributions with variances \( \hat{\sigma}^2_r, \hat{\sigma}^2_v, \) and \( \hat{\sigma}^2_w \) given by (32), (34), and (35) respectively. We see that the variance along the radial direction \( \hat{r} \) is larger than those in the two perpendicular directions. This difference is similar to that for the variances of the three components of TRE observed by Fitzpatrick et al. in 2001 [5], except that in the case of TRE the variance along \( \hat{r} \) is smaller than the variances in the two perpendicular directions. The FRE\(^2 \) at any fiducial marker is shown to be distributed as the sum of three chi-square
variables with variances $\sigma_r^2$, $\sigma_v^2$, and $\sigma_w^2$ as shown in (44). We also derived the distribution of FRE along any arbitrary direction. It is shown to be a normal distribution with zero mean and variance given by (43).

We also verified that the approximate distribution for individual FRE that we derived works well for practical cases by means of simulations. We showed that this predicted distribution matches closely the results of simulation based on the point-based registration algorithm. We also compared the predicted variances along the three orthogonal directions $\mathbf{r}$, $\mathbf{v}$, and $\mathbf{w}$, and any arbitrary direction $\mathbf{a}$ with the variances that were obtained by simulations based on point-based registration algorithm that reflect typical situations, and found that they match very closely.

It is important to note though that the predicted distribution is just an approximation up to first order. It assumes that the second order term of the rotation matrix $R^{(2)}$ and other higher order terms are negligible. When this second order term becomes big, then our approximation will not work. $R^{(2)}$ can be expected to be negligible as long as $\langle \text{FLE}^2 \rangle$ is small compared to each of the second moments of the fiducial configuration about its principal axes [5]. This requirement holds for all our simulated configurations (smallest second moment $= 107 \times \langle \text{FLE}^2 \rangle$), and for all reasonable fiducial configurations, which may be planar but not collinear, and for all reasonable FLEs. It may not hold though, when the fiducials are arranged in a near-collinear configuration and FLE is large.

6. CONCLUSION

We have derived the first approximate expressions for the distribution of FRE of an individual fiducial marker $\mathbf{r}$. They agree very closely to ground-truth simulations of point-based registration when the fiducial configuration is non-collinear and FLE is at a level typical of surgical applications, showing that our first-order approximation holds good.

Our derivation assumes equal weighting for all fiducials, but it could be extended to include unequal weighting. The main assumption in our derivation is that FLE for all the fiducials is independent and drawn from a zero-mean normal distribution with equal variance for all fiducials in all directions, i.e. the FLE distribution is the same for all fiducials and is isotropic. This assumption is common in the analysis of errors in this field [4, 5, 9], and results based on this assumption have proven quite useful in the field of neurosurgery, orthopedic surgery, and radiation oncology where bone-attached fiducial markers are being used. It must be noted though that FLE may be anisotropic in image space when the spatial resolutions in the image coordinate directions are different and anisotropic in the physical space when coordinate reference frames are used [15]. The problem of extending the derivations to include this anisotropy is difficult with no closed-form solution to the rigid-body point-based registration problem with different weights in different directions. Currently available solutions are iterative in nature [16, 17] and hence time consuming. With the necessity to perform the registration with high speed, it is preferred to assume isotropic FLE and use the closed-form solution for the registration. Wiles et al. [17] recently provided a statistical model for TRE when FLE is anisotropic but still used the closed-form solution for the registration purpose. It should be possible to extend the statistical model for FRE also using a similar approach.

The formulas derived in this paper can be used to understand better the accuracy of a fiducial system. The goal of any fiducial system is to register two spaces so as to produce the smallest possible TRE at the intended target regions. Knowledge of the distribution of the individual FRE of a fiducial marker could guide us find fiducials that are damaged and thereby increase TRE. Once they are identified those fiducials could be ignored or weighted less than the other markers in order to improve the TRE of the system. It is hoped therefore that our efforts will result in improved registration accuracy and better outcomes when image guidance is used in surgical procedures.

REFERENCES


