A Technique for Accurate Magnetic Resonance Imaging in the Presence of Field Inhomogeneities

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Abstract—Static field inhomogeneity in magnetic resonance (MR) imaging produces geometrical distortions and distortions in intensity. The inhomogeneity may be caused either by imperfection in the magnet system or by magnetization of the object being imaged. In this paper we present a technique for producing geometrically accurate MR images with undistorted intensity in the face of high levels of static field inhomogeneity arising from either source. The technique requires the acquisition of two images of the same object with altered gradients. Based on a knowledge of these gradients it employs an automatic postprocessing step that exploits some invariant characteristics of the distortions to produce a rectified image from the two acquired images. No phantom imaging is involved and no operator interaction is required. We present a theoretical justification of the technique and compare it to other techniques, and we present experimental results that show that the technique works. The improved accuracy in geometry and intensity may improve the reliability of stereotactic surgery, may enhance the feasibility of both clinical and industrial imaging via external fields, and may increase the resolution of microscopic imaging.

I. INTRODUCTION

The presence of static field inhomogeneities in magnetic resonance imaging (MRI) produces geometrical distortions and distortions in intensity. The problem arises because of the necessity to impress highly linear gradients on a highly uniform static field in order to encode the positions of nuclei by means of linear spatial variation in their frequency of precession. It is typically assumed for the purpose of image reconstruction that the composite field, composed of both the impressed gradient and the static field, exhibits linear variation. Any inhomogeneity in the static field, whether it originates from system imperfection or from diamagnetism or paramagnetism of the object being imaged, will corrupt the spatial encoding of the nuclei. The resulting image will suffer from both geometrical and intensity distortion. The effect is important both in macroscopic and microscopic imaging. In macroscopic imaging geometrical distortions from 3 to 5 mm have been reported from phantom studies as well as theoretical analysis for clinically tuned systems [5],[15],[24]. This accuracy is unacceptable for applications such as neuro-

stereotactic surgery where images are used to determine the spatial location of a target object and one millimeter accuracy is required. Much larger errors can be expected in the area of industrial nondestructive testing when objects too large to be placed within a coil must be imaged in an external field or when the objects to be imaged contain or are attached to ferromagnetic materials. A similar situation will occur in medical imaging if a patient is imaged in an external field in order to afford access during procedures. In microscopic imaging intensity distortion results in a fundamental limitation on the attainable resolution [2]. This paper proposes a novel technique for producing geometrically accurate images with nondistorted intensity in the face of large field inhomogeneities. We call the resulting images "rectified" images. The technique requires the acquisition of two images of the same object with altered gradients. It employs an automatic postprocessing step that exploits some invariant characteristics of the distortions to produce a rectified image from the two acquired images. No phantom imaging is involved and no operator interaction is required. We provide theoretical justification for the technique based on a standard spin-echo sequence, and we show the results of experiments to demonstrate that the technique works. In Section II we derive the form of the distortion and consider its magnitude. In Section III we review previously proposed techniques. In Section IV we describe our proposed technique. In Section V we describe our experiments and discuss the results. In Section VI we conclude with a summary and some suggestions for future work.

II. THE FORM OF THE DISTORTION

We consider a multislice spin-echo image. While neither the particular form of the pulse sequence nor the nuclear species is crucial to the technique, we will for concreteness consider a typical sequence as depicted in Fig. 1 applied to protons. In this sequence, $G_z$ is the readout gradient, $G_x$ is the phase encoding gradient, and $G_y$ is the slice selection gradient. In our analysis we consider the form of the signal after its high frequency component $\gamma B_0$ has been removed. To simplify our analysis we ignore the time dependence of the signal during readout due to $T_1$ and $T_2$ decay, and we ignore the finite thickness of the slice. We define a density function $\rho(x,y,z)$ that denotes the number of excited spins per unit volume in the object to be imaged. The value of this density function at a position $x,y,z$ depends on the proton density, $T_1$, $T_2$, and
followed immediately by a negative refocusing gradient to cancel the phase
place while the data gate is high. The positive
dispersion within the slice produced by the positive gradient. A nonselective
readout in the
effected in the y direction by means of the gradient
applied in the presence of the positive gradient
direction by means of a selective 90° RF pulse at an angular frequency
Fig. 1. A spin-echo pulse sequence. Slice selection is effected in the
nylomagnetic ratio for the proton. The signal during the time
would select a slice at
the central angular frequency of the selective pulse,
nominal angular frequency in the absence of the gradients,
i.e.,
180° RF pulse is applied at
interval, TE
inhomogeneity have the form
where \(A\) is some constant of proportionality determined by
the geometry and sensitivity of RF coils. We have assumed that the
transformation is one-to-one (see the end of Section IV), and
therefore that the Jacobian is finite and nonzero. We can now
integrate over \(z'\) to get
\[
s(t, k, z_1) = - \frac{B_e(x, y, z)}{G_x}
\]
where
\[
z_1 = z + \frac{B_e(x, y, z)}{G_x}
\]
A Fourier transform of \(s(t, k, z_1)\) over \(\tau\) and \(k\) constructs the
distorted image,
\[
i_1(x_1, y_1, z_1) = -A\rho(x, y, z)/J(x_1, y_1, z_1)
\]
(4)
where \((x, y, z)\) is the accurate image that would be obtained
if there were no inhomogeneity. There are two components to
the distortion in the image \(i_1\)—the geometrical distortion given by
(1) and (3), and the intensity distortion manifested by the
division by the Jacobian in (4). Equations (1) and (3) show
that at points where \(B_e\) is nonzero, the image is displaced from
\(x, y, z\) to \(x_1, y_1, z_1\). The ratios among the \(x, y,\) and \(z\)
components of the displacements are constants throughout the
image. In particular, \(y_1 - y\) equals zero and \(z_1 - z\) equals

\[k \equiv -G_y t_y.
\]
Because of these constant ratios the Jacobian has the simple form,
\[
J(x_1, y_1, z_1) = 1 + \frac{1}{G_x} \frac{\partial B_x(x, y, z)}{\partial x} + \frac{1}{G_z} \frac{\partial B_z(x, y, z)}{\partial z}.
\] (5)

From (1), (3), and (5) it can be seen that the distortions can be reduced by enlarging the magnitudes of the readout and slice selection gradients. Increasing these gradients has unfortunate side effects, however, including increased eddy currents and, for the readout gradient, decreased signal-to-noise ratio. Careful shimming provides the level of field homogeneity necessary for images with geometrical distortions below 1 mm considered reliable [13], [18, p. 201].

Even with an ideal magnet the local field will be perturbed by the magnetization of the object being imaged. The magnitudes of the induced distortions have been carefully studied [2], [4], [10], [15], [21]. As a simple example, an air-filled infinite homogeneous cylindrical medium immersed in water has been analyzed in [4], [15], and [21]. When the axis of the cylinder is orthogonal to the axis of \( B_0 \), there will be an extra field induced by the cylinder, whose magnitude inside the cylinder is given to a first approximation [4] by
\[
|B_x| \approx 3 \chi_w - \chi_a |B_0|/6
\]
where \( \chi_w \approx -0.00001 \) is the magnetic susceptibility of water and \( \chi_a = 0.0 \) is the magnetic susceptibility of air. For this case we find that \( |B_x| \approx 5 \) ppm. For a 1.5 T system with a readout gradient of 2.5 mT/m (1) shows a geometrical distortion of 3 mm in the readout direction. Distortions of this magnitude can easily be observed in phantoms and are often suspected in clinical cases. Shifts of approximately this size are apparent in our phantom experiments (see Section V).

Because the substance being imaged is rarely homogeneous, the spatially varying effects are typically nonlinear and therefore will cause errors in measuring position even when the measurements are made relative to sets of fiducial marks in the image, as in the case of stereotactic surgery. The magnitude of distortion due to susceptibility is proportional to the strength of the static field [2], [12], [15]. Hence, it can be expected that as the field gets larger, the magnetic susceptibility contribution to the geometrical distortion problem will become more noticeable. For example, by increasing the main field from 1.5 to 4 T and holding all other parameters unchanged, we increase the distortion magnitude shown in the above example from 3 to 8 mm.

III. PREVIOUS METHODS

Previous methods to reduce geometrical distortion may be divided into the following two categories: those using phase encoding to avoid distortions and those that are based either directly or indirectly on independently measured distortion field maps.

A. Phase Encoding Based Methods

As (2) shows, phase encoding is insensitive to field inhomogeneity. Several methods have been proposed for taking advantage of this insensitivity. Unfortunately, these techniques require long imaging times because they sacrifice the inherent parallel data gathering power provided by frequency encoding in favor of the immunity to field inhomogeneity provided by phase encoding. Examples of techniques relying on phase encoding are provided by Bendel's "echo projection imaging" [1], Miller and Garway's "refocused gradient imaging" [17], and a method of filtered backprojection recently proposed by Callaghan that includes a sequence of gradient reversals [2]. These techniques all use alternating gradients during a sequence of 180° pulses. There is a lower limit to the length of the pulses required to produce accurate natures. As a result despite the parallelism the imaging times are long.

B. Distortion Field Map Based Methods

In another category the focus is to acquire images of a phantom of known shape to derive a map of the distortion field. For reconstruction from projections Lai [12] proposed a curvilinear back projection technique for large inhomogeneities. Although the limit of inhomogeneity that his technique can handle was not presented, the computer simulation results indicate that image blurring accompanied by the distortion can be largely avoided. It is very sensitive to errors in estimates of field inhomogeneities as pointed out by others [20].

For Fourier transform imaging methods Hutchison [8] proposed a method to remove intensity distortion. Using the distortion field obtained from an image of a uniform phantom his technique uses the phase information obtained from the phantom image to calibrate the object image. A major drawback of his method is the lack of correction for geometric distortion. O'Donnell and Edelstein [20] introduced their method for coping with field inhomogeneity and gradient non-linearity using the spin warp method. Assuming that the field map is somehow known, they show that the spin warp method is then much more robust in handling inhomogeneities than Lai's correction method for projection imaging. Schmitt and Schad [25], [24] proposed a variation of the method of O'Donnell and Edelstein by including a polynomial approximation for the transformation established by physical information from a set of points and their locations in the image. The effectiveness of the method was illustrated by a two-dimensional phantom image displaying pin-cushion artifacts due to the existence of field inhomogeneity and gradient nonlinearity. After applying the method, the distortion was largely removed in the image. Sekihara [22], [23] presented a similar method. His method involves measuring the field map using a pulse sequence incorporating two phase encoding directions with a uniform phantom, collecting an image from the object, and rectifying the object image by applying the knowledge of map, taking into consideration both the geometrical and intensity components of the distortions.

Other techniques in this category focus more on how the distortion field is measured. Using a phantom with an array of
point objects, Kawanaka [10], [11] proposed exchanging the role of phase and readout gradients to obtain the distortion in terms of the correspondence between the physical points and their images. The displacements of the objects are approximated by polynomials. The coefficients of the polynomial estimated by a least-squares method are then used to map the perturbed magnetic field. The method measures not only static field inhomogeneity but also gradient nonlinearities. Yamamoto proposed a method [30] in which two images of a phantom are acquired with opposite gradients. Using a phantom that consists of an array of point objects he showed that by establishing a correspondence between the positions of these points in the two images it is possible to obtain a field map.

These methods suffer from some or all of the following disadvantages. 1) It is not feasible to make a phantom that duplicates the biological system to be imaged. In fact, the phantoms proposed for use are all uniform, giving no consideration to variations in the field produced by the magnetic susceptibility of the object being imaged. Hence a distortion correction method based on field maps obtained from uniform phantoms will compensate only for field distortions caused by sources other than the imaged object. 2) Even for distortions caused by sources other than the image object, because these sources may vary over time, the field map acquired with the phantom may be different from that which obtains during the acquisition of the patient image [6]. 3) If the field map is obtained by interpolating from values measured at certain fiducial locations, for positions that are not near the fiducials the interpolation and extrapolation techniques used will limit the accuracy of the correction process.

More recently, Jonckheere et al. have proposed a method specifically devised to correct for geometrical positioning errors caused by the magnetic susceptibility of the aluminum base ring in stereotactic surgery [9]. This technique requires that the source of the magnetic disturbance be outside the region to be corrected, as is the base ring. It also requires that the positions of a special ring of fiducial markers be known, that this marker ring surround the critical region, and that the source of the disturbing field be outside the marker ring. It assumes that the form of \( B_z(x, y, z) \) can be interpolated by a polynomial of finite degree, in this case quadratic. The benefit of this technique is that it does not rely on comparing images acquired at different times because the phantom is present in the patient image. Its disadvantages are its assumption of the polynomial form of \( B_z \) and its restrictions on the positioning of the source of the magnetic disturbance.

Another class of methods may, by using a special phase imaging pulse sequence to map the distortion field, circumvent some of the above problems. Phase imaging provides measurement across the whole image and hence does not suffer from the drawbacks related to interpolation/extrapolation. Proposed methods range from imaging the relative field perturbation [28] for qualitative field homogeneity inspection to patient dependent susceptibility mapping [16], [27], [26], [19]. However, using this method implies that for each patient study there should be an extra image taken to take into account the patient induced field inhomogeneity. As a remedy for this drawback, Feig et al. [6] proposed a technique that combines inhomogeneity mapping with patient imaging by extracting the inhomogeneity information from the phase shifts of the patient image itself. By postprocessing the phase information from the patient image, his method is capable of correcting the distortions. However, he recommends against its application to multi-slice SE volumetric images because it cannot handle distortion in the slice selection direction. Sekihara's idea [22] (cited above) is similar to Feig's but with an approximation in the determination of the inhomogeneity mapping.

IV. PROPOSED TECHNIQUE

Our technique involves the acquisition of two images using identical pulse sequences except for changes in the gradients. We combine these two distorted images to produce a rectified image. In this section we first present an approximate technique for rectifying single slice images. Then we present an exact three dimensional technique for rectifying volume images. Finally we consider the size of the inhomogeneity that our technique can accommodate.

A. Approximate Rectification of Single Slice Images

So far we have treated all images as extending over three dimensional volumes. These images are typically collections of contiguous sets of two dimensional slices images, each acquired with a sequence such as that depicted in Fig. 1. It is possible, however, to rectify the \( x \) distortion in a slice image approximately by acquiring and combining only two single slices. The acquisitions are identical except that the preparation and readout gradients are changed. From (3) we see that the \( z \) displacement is identical in two such images, which we denote as \( i_1(x_1, y, z_1) \) and \( i_2(x_2, y, z_1) \). Because \( y \) and \( z_1 \) are the same in each image, we can for convenience omit both \( y \) and \( z_1 \) dependence in our equations. These two images are acquired from the same, but not necessarily planar, slice. The shape of this slice is given by (3). That shape will not be corrected by our approximate method, but the intraslice distortion will be approximately corrected. We show how the degree of the approximation is affected by the sizes of the gradients and the inhomogeneity below.

The intraslice distortion is confined to that given by (1), which with the \( y \) and \( z_1 \) dependencies suppressed has the form,

\[
x_1 = x + \frac{B_z(x)}{G_x}.
\]

We assume that \( G_z \) is sufficiently large to make the following inequality hold at all points in the slice:

\[
\frac{1}{G_z} \frac{\partial B_z}{\partial x} \ll 1.
\]

The expression for the Jacobians (5), then becomes

\[
J(x_1) = 1 + \frac{1}{G_x} \frac{dB_z(x)}{dx} = \frac{dx_1}{dx}.
\]
where we have used (6) to derive the second equality. We note that for each value of $y$, (6) and (8) hold for all values of $x$. We may thus treat the pair of two-dimensional images as a collection of independent pairs of one-dimensional images, one for each value of $y$. Each such one-dimensional image pair may be processed independently to produce a rectified one-dimensional image. The resulting collection of one-dimensional images makes up a single, rectified, two-dimensional image. The problem of producing a rectified two-dimensional image thus reduces to the problem of producing a rectified one-dimensional image.

We acquire the first slice image with the sequence shown in Fig. 1. We refer to the one dimensional image at $y$ as $i_1(x_1)$. We acquire the second slice with both the preparation and readout gradients changed to $G_x/\alpha$ where $\alpha$ is some signed number. We refer to the second one-dimensional image at $y$ as $i_2(x_2)$. Our task is to determine the rectified one-dimensional image $i(x)$ at $y$ when given these two distorted images. Fig. 2 illustrates our approach. A small perturbation to the presence of the perturbation, rather than its prescribed

$$B_0 + B_0(x) + zG_x :$$

where $z$ is some signed frequency that is proportional to $x$. We refer to the second one-dimensional image at $y$ as $i_2(x_2)$. The problem of producing a rectified two-dimensional image thus reduces to the problem of producing a rectified one-dimensional image.

We acquire the first slice image with the sequence shown in Fig. 1. We refer to the one dimensional image at $y$ as $i_1(x_1)$. We acquire the second slice with both the preparation and readout gradients changed to $G_x/\alpha$ where $\alpha$ is some signed number. We refer to the second one-dimensional image at $y$ as $i_2(x_2)$. Our task is to determine the rectified one-dimensional image $i(x)$ at $y$ when given these two distorted images. Fig. 2 illustrates our approach. A small perturbation $B_0(x)$ of the uniform field $B_0$ is shown in Fig. 2(a). When a linear gradient $G_x$ is turned on, the total field then becomes $B_0 + B_0(x) + xG_x$ at position $x$. As is demonstrated in Fig. 2(b), a spin isochromat at location $x_0$ will possess a precessional frequency that is proportional to $B_0 + B_0(x_0) + x_0G_x$ due to the presence of the perturbation, rather than its prescribed value: $B_0 + x_0G_x$. By noting that $B_0 + B_0(x_0) + x_0G_x$ equals in magnitude $B_0 + x_1G_x$ after reconstructing an image without the knowledge of $B_0(x)$, it can be seen that the spin isochromat at $x_0$ will appear in the image as though it were at $x_1$. As Fig. 2(c) shows, using a different gradient causes the spin isochromat at $x_0$ to appear as though it were at $x_2$. The displacements $\Delta x_1 \equiv x_1 - x$ and $\Delta x_2 \equiv x_2 - x$ depend on $G_x$ and $\alpha$. Note for the special case $\alpha = -1$, as in this example, that $\Delta x_2 = -\Delta x_1$. Thus, $x$ is equidistant from $x_1$ and $x_2$. This figure illustrates the basis for our method: the distortion may be manipulated by adjusting the gradient.

We can obtain an equation relating positions in the second image $i_2(x_2)$ to positions in the undistorted image by substituting $x_2$ in place of $x_1$ and $G_x/\alpha$ in place of $G_x$ in (6) to get

$$x_2 = x + \alpha B_0(x)/G_x. \quad (9)$$

Solving (6) and (9) for $x$ we find that

$$x = (\alpha x_1 - x_2)/(\alpha - 1), \quad (10)$$

which for the special case $\alpha = -1$ illustrated in Fig. 2 reduces to $x = (x_1 + x_2)/2$. In some cases it might be possible to identify a few corresponding points in $i_1(x_1)$ and $i_2(x_2)$ interactively, as in the phantom employed by Yamamoto [30]. We could then easily determine their true positions from (10). However, except for easily identifiable point objects in the images, such a correspondence is not readily available. Fortunately, we can determine these correspondences automatically without the use of phantoms as follows. We note that for this one dimensional case (4) can be written using (8) as

$$i_1(x_1) = i(x_1)/dx_1.$$  

Thus, the two distorted images are related to the undistorted image as follows:

$$i_1(x_1) dx_1 = i(x), \quad i_2(x_2) dx_2 = i(x). \quad (11)$$

Applying the chain rule of differentiation to these two equations we find that

$$dx_2 = i_1(x_1)/i_2(x_2). \quad (12)$$

Equation (12) can be solved numerically with a proper boundary condition to give $x_2$ in image $i_2$ as a function of $x_1$ in image $i_1$. With such a correspondence available, (10) yields accurate spatial information.

The boundary condition is achievable if we can identify one $x_{20}$ corresponding to some $x_{10}$. Starting there we can integrate (12) to get $x_2$ for every $x_1$. Such a condition is available at the boundary between the imaged object and the background, provided the imaged object does not occupy the entire field of view. If an error is made in determining this boundary, however, the error will propagate along the $x$ axis. To investigate the nature of this propagation it helps to convert (12) from a differential equation to an equivalent integral equation. We can do that by multiplying both sides by $i_2(x_2)$, integrating both sides over $x_1$, and making a change of integration variable from $x_1$ to $x_2$ on the left to get

$$\int i_2(x_2) dx_2 = \int i_1(x_1) dx_1.$$  

We convert the indefinite integrals to definite integrals by changing integration variables, $x_1 \rightarrow \xi_1$ and $x_2 \rightarrow \xi_2$, and integrating from $x_{10}$ to $x_1$ and from $x_{20}$ to $x_2$

$$\int_{x_{20}}^{x_2} i_2(\xi_2) d\xi_2 = \int_{x_{10}}^{x_1} i_1(\xi_1) d\xi_1. \quad (13)$$

For a given initial pair $x_{10}, x_{20}$, solving (12) is equivalent to finding $x_2$ for every $x_1$ so that the integrals remain equal in
We subdivide the integral on the left side of (13). We note that if an erroneous boundary point \( x'_{20} \) is paired with \( x_{10} \) then an erroneous point \( x'_{2} \) will be paired with \( x_{1}, \)

\[
\int_{x'_{20}}^{x'_{2}} i_{2}(\xi_{2})d\xi_{2} = \int_{x_{20}}^{x_{2}} i_{1}(\xi_{1})d\xi_{1}.
\]

We define the erroneous shift \( \Delta_{2} \equiv x'_{2} - x_{2} \), which is the result of the propagation of the initial shift \( \Delta_{20} \equiv x'_{20} - x_{20} \). We subdivide the integral on the left side of (14) as follows:

\[
\int_{x_{2}}^{x_{2} + \Delta_{2}} i_{2}(\xi_{2})d\xi_{2} = \int_{x_{20}}^{x_{20} + \Delta_{20}} i_{2}(\xi_{2})d\xi_{2}.
\]

Making use of (13) and reversing the direction of integration in the first integral yields

\[
\int_{x_{2}}^{x_{2} + \Delta_{2}} i_{2}(\xi_{2})d\xi_{2} = \int_{x_{20}}^{x_{20} + \Delta_{20}} i_{2}(\xi_{2})d\xi_{2}.
\]

We can draw three conclusions from this equation. 1) Since image intensity is always positive, \( \Delta_{20} \) and \( \Delta_{2} \) must have the same sign. 2) \( |\Delta_{2}| \) will tend to be smaller at \( x_{2} \) if the intensity is larger in the vicinity around \( x_{2} \). 3) \( |\Delta_{2}| \) will tend to be smaller at all \( x_{2} \) if the intensity is smaller in the vicinity of the boundary. The boundary position will typically be defined as that position at which the intensity first rises above some chosen threshold. When the boundary is chosen to be the interface between the background and the imaged object, the threshold can in principle be chosen to be zero since there is no signal generated outside the object. Because of the presence of noise a positive threshold is required, but it need be no larger than the largest expected value of the noise. At the object-background interface we can expect the intensity gradient to be large. As in any application that uses a threshold to define a boundary, a large intensity gradient will reduce the tendency of additive noise to shift the boundary. The third conclusion above reveals, however, an interesting additional benefit in this particular application deriving from the fact that the threshold is of the order of the noise. It is illuminated by a simple approximation: Since the shift might be expected to be small, it is reasonable to make the approximation that the intensity is constant over the integrals in (15).

This approximation yields \( i_{2}(x_{2})\Delta_{2} \approx i_{2}(x_{20})\Delta_{20} \) or equivalently,

\[
\Delta_{2} \approx \left[ T/i_{2}(x_{2}) \right] \Delta_{20}
\]

where we have used the fact that \( i_{2}(x_{20}) \) will be approximately equal to the threshold \( T \). Since the threshold will be set equal to the noise level, we see from (16) that the size of the propagated error at \( x_{2} \) will remain below that of the initial error unless the intensity in the vicinity of \( x_{2} \) drops below the noise.

To calculate the image intensity at \( x \), we proceed as follows. Taking the derivative of both sides of (10) with respect to \( x \) and substituting from (11), we find that

\[
i(x) = (1 - \alpha) \frac{i_{1}(x)}{i_{1}(x)} i_{2}(x) \cdot \frac{i_{1}(x)}{i_{1}(x)} - \alpha i_{2}(x).
\]

Equations (10) and (17) give the solution for a gradient ratio of \( \alpha \). For \( \alpha = -1 \) the solution becomes

\[
x = (x_{1} + x_{2})/2
\]

and

\[
i(x) = \frac{2i_{1}(x)}{i_{1}(x) + i_{2}(x)}.
\]

We note here that any error in determining \( x_{2} \) for a given \( x_{1} \) will result in an error in the image intensity calculation. If \( x_{2} \) is erroneously shifted to \( x_{2}' \) by errors in the solution of (12), whether caused by numerical approximations or by the error in boundary location discussed above, the calculation of \( i(x) \) will be based on intensities arising from positions between two different anatomical positions. The result will be a blurring of \( i(x) \).

We have derived, subject to our approximation, the rectified one-dimensional image at \( y \). To produce an approximately rectified two-dimensional image we repeat this derivation for all values of \( y \) and assemble the resultant set of one-dimensional images into the two-dimensional image \( i(x, y, z) \). Note that in our above analysis we did not assume any source for the distortion field. Be it from the main field inhomogeneity, object magnetization, or both, the change in the gradient causes a change in the geometrical and intensity distortions, which are reflected in the image as shifts of objects and alterations in intensity.

The only approximation that we have made above is to ignore the third term of the Jacobian, the derivative with respect to \( z \), in (5). (This approximation is removed in our full, three-dimensional solution below.) In order to examine our approximation we will reinstate the \( z \) dependence in the argument lists. For notational convenience we define \( e_{e}(x, z) \) and \( e_{y}(x, z) \) such that

\[
\frac{1}{G_{x}} \frac{\partial B_{x}(x, z)}{\partial x} = e_{e}(x, z),
\]

\[
\frac{1}{G_{z}} \frac{\partial B_{z}(x, z)}{\partial z} = e_{y}(x, z)
\]

where \( \epsilon \) is a positive, unitless, adjustable factor for varying the sizes of these two terms of the Jacobian. The two distorted images can be related to the undistorted image by using (4) and (5) with \( G_{x} \rightarrow G_{x}/\alpha \) for \( i_{2}(x_{2}, z) \) as follows:

\[
[1 + \epsilon e_{e}(x, z) + \epsilon e_{z}(x, z)]i_{1}(x_{1}, z) = i(x, z),
\]

\[
[1 + \alpha \epsilon e_{e}(x, z) + \epsilon e_{z}(x, z)]i_{2}(x_{2}, z) = i(x, z).
\]

Reorganizing (19) and using (8) gives

\[
\frac{\partial x_{1}}{\partial x} i_{1}(x_{1}, z) = \frac{1 + \epsilon e_{e}}{1 + \epsilon e_{e} + \epsilon e_{z}} i(x, z),
\]

\[
\frac{\partial x_{2}}{\partial x} i_{2}(x_{2}, z) = \frac{1 + \alpha \epsilon e_{e}}{1 + \alpha \epsilon e_{e} + \epsilon e_{z}} i(x, z)
\]
where we have suppressed the explicit $x, z$ dependence of $e_x$ and $e_z$ on the right side. In our one-dimensional rectification, we assumed that the left sides of (20) and (21) were equal, by virtue of (11). The relative error between the left sides of (20) and (21), normalized by the left side of 20, equals \((1 - \alpha) e_x e_z e \epsilon^2 / ([1 + e_x e_z / (1 + \alpha e_x + e_z)]),\) which approaches \((1 - \alpha) e_x e_z e \epsilon^2\) as \(\epsilon \to 0.\) Thus, the one-dimensional rectification method is accurate to second order in \(\epsilon.\) We note that \(\epsilon\) measures only the spatial derivatives of \(B_x,\) not \(B_e\) itself. Thus, we find that the accuracy of the approximation depends not on the magnitude of the deviation from \(B_0,\) but only on its spatial variation. This superlinear increase in accuracy of the approximation with a decrease in the spatial variation of the inhomogeneity makes it feasible to apply our rectification technique exclusively within a slice, despite the concomitant presence of distortions in the slice selection direction, as long as the spatial variation of the inhomogeneity is not too large.

### B. Exact Three-Dimensional Rectification

We now consider volume images, which are sets of contiguous slices acquired with a sequence such as that from Fig. 1. The extension to three dimensions requires that we consider the distortion in the $z$ direction, (3), that we include the third term in (5), and that we include all spatial variables in the argument lists. We begin by defining three vectors,

\[ g \equiv (g_x, g_y, g_z) \equiv (1/G_x, 0, 1/G_z), \]

\[ r \equiv (x, y, z), \]

\[ r_1 \equiv (x_1, y_1, z_1). \]  

(22)

These vectors permit us to represent the geometrical distortion expressed by (1), (2), and (3) concisely as $r_1 = r + B_e(r)g$. Fig. 3 illustrates the form of the geometrical distortion using three sample spin isochromats as examples. Note that the vectors $B_e(r)g$ are parallel for all isochromats throughout the volume image. We next define a new coordinate system, $(x_1, y_1, z_1$), rotated about the $y$ axis such that the $x_r$ axis lies along the vector $g$ and the $y_r$ and $y$ axes coincide. The new coordinate system is indicated by the dashed-lines in Fig. 3. In this system the geometrical distortion has the form

\[ x_{r1} = x_r + B_e(x_r, y_r, z_r)g, \]  

(23)

\[ y_{r1} = y_r, \]  

(24)

\[ z_{r1} = z_r \]  

(25)

where $g \equiv |g| = \sqrt{1/G_x^2 + 1/G_z^2}$. Equations (23) through (25) show that all displacements occur along the $x_r$ axis. The Jacobian in this system has the simple form

\[ J(x_{r1}, y_{r1}, z_{r1}) = 1 + g \frac{\partial B_e(x_r, y_r, z_r)}{\partial x_r} \frac{dx_{r1}}{dx}, \]  

(26)

and (4) is transformed to

\[ i_{r1}(x_{r1}, y_r, z_r) \frac{dx_{r1}}{dx} = i_r(x_r, y_r, z_r), \]  

(27)

This transformation to the rotated coordinate system reduces the three-dimensional distortion problem into a one-dimensional problem. To produce a rectified image we acquire a second image as we did for the approximate rectification, but this time we alter not only the preparation and the readout gradients, but also the slice selection gradient, changing them from $G_x$ and $G_z$ to $G_x/\alpha$ and $G_z/\alpha$. The distortion in this second image will also occur only along the $x_r$ axis. It will have the form

\[ x_{r2} = x_r + \alpha B_e(x_r, y_r, z_r)g. \]

Now to produce a rectified image we employ the same postprocessing that we used in the approximate case. We drop the subscript $r$, suppress the $y$ and $z$ dependencies and use (10), (12), and (17).

### C. Limits of Distortion

Equations (26) and (27) show that the size of the inhomogeneities that this technique can accommodate is limited by the requirement that the Jacobian be non-zero. This requirement is equivalent to the statement that the mapping from one image space to another must remain one-to-one. In regions for which the mapping is not one-to-one there will be an overlap. The signal from two different points in object space will add together into one point in image space.

We must require, therefore, from (26) in the rotated coordinate system, suppressing the variables, $y_{r1}$ and $z_{r1}$ and dropping the subscript $r$, that $J(x_1) = 1 + g\partial B_e(x)/\partial x > 0$ for the first image and $J(x_2) = 1 + \alpha g\partial B_e(x)/\partial x > 0$ for the second image. Thus, we require that min $(dB_e(x)/dx, \alpha dB_e(x)/dx) > -1/g$. We consider the special case $\alpha = -1$ in our experiments. Using this value and the definition of $g$ gives $|dB_e(x)/dx < |G_xG_z|/\sqrt{G_x^2 + G_z^2}$. For a 1.5 T imager and 1.57 mT/m gradients, the smallest gradients used in our experiments, this restriction is met for
a spatial variation in inhomogeneity as high as 1.05 ppm per 
mm, which means a maximum ±105 ppm variation over a 
field of view of 20 cm in its longest dimension, which is many 
times the typical inhomogeneity expected in a commercial 
system. With larger gradients the restriction is obeyed for 
proportionally larger inhomogeneities.

V. EXPERIMENTAL RESULTS

In our theoretical development we described both an approx-
imate technique for single slice images and an exact technique 
for volume images. We present here an experimental test of 
the approximate technique. Our results below indicate that the 
approximation is excellent. Because this technique does not 
require the acquisition of volume images and because it is 
simpler to implement, we believe it to be the more clinically 
applicable method. A test of the exact technique on volume 
images can be found in [3] where it is shown to work even in 
the presence of very large inhomogeneities. All images were 
acquired with the spin-echo sequence of Fig. 1 on a Siemens 
1.5 T Magnetom imager housed in the Vanderbilt University 
Medical Center. The image reconstruction was effected by the 
Siemens system, and the modulus of each complex image was 
transferred via magnetic tape to a Sun Microsystems 3/260 
workstation for post processing. Two image sets were acquired 
in each case with gradients altered between acquisitions. In 
all cases the ratio $\alpha = -1$ was used, which means that the 
gradients were simply reversed with no change in magnitude.

The post processing consisted of the direct application of 
the techniques we have described. Before the technique was 
applied the images acquired with the reversed gradients were 
mirrored in the direction of the gradient reversal to bring the 
$\vec{x}_1, \vec{y}_1, \vec{z}_1$ and $\vec{x}_2, \vec{y}_2, \vec{z}_2$ coordinates into the same orientation. 
Then two numerical techniques were tested: a fourth-order 
Runge-Kutta solution of (12), (7) and a direct solution of 
(13), [3]. Each technique was tested for noise sensitivity and 
found to be stable [3]. The differences in the processed images 
produced by the two different numerical techniques were 
negligible. As discussed above, the integration requires that 
one pair of corresponding points be established on the basis 
of some a priori information. We have chosen the interface 
between air and the boundary of the object being imaged, 
which is in every case clearly visible in both image sets, 
to establish the correspondence. This boundary is established 
by searching from one edge of the image along the readout 
direction for the first pixel whose image intensity is above a 
threshold. The threshold is chosen as low as possible subject 
to the requirement that it be above the noise. This threshold, 
below which pixels are defined to be part of the background, is 
the only arbitrarily adjustable parameter in our technique. We 
have found that image intensity rises so rapidly at the interface 
between air and object that such a threshold can easily be 
chosen and that the processed images exhibit little sensitivity 
to its value.

We describe two experiments to show that the approximate 
technique corrects both geometrical and intensity distortions. 
The subject of our first experiment is a grid, whose images 
are shown in Fig. 4. This phantom is a two dimensional 
array of orthogonally interlocking plastic strips immersed in a 
4 mM CuSO$_4$ solution. This phantom provides a convenient 
model for demonstrating geometrical distortions because the 
plastic strips are straight. Any deviation in the image from 
straightness indicates a geometrical distortion. The imaging 
parameters are $TE = 22.2$ ms, $TR = 300$ ms, readout gradient 
$= 2.61$ mT/m, slice selection gradient $= 1.8$ mT/m, and slice 
thickness $= 10$ mm. Pixel size is $1.17$ mm. Fig. 4(a) and 
(b) show the unprocessed images. Fig. 4(a) with the readout 
gradient directed from right to left and Fig. 4(b) with the 
readout gradient directed from left to right. The phase encoding 
gradient was vertical. The phase encoding and slice selection 
gradients were unchanged for both images. The bowing to 
the left in Fig. 4(a) and the bowing to the right in Fig. 4(b) 
are the geometrical distortions. Equation (2) indicates that 
there should be no geometrical distortion along the phase 
encoding direction. The straightness of the horizontal lines 
in these images indicates that there is indeed no distortion 
in this direction. The two white, vertical lines have been 
superimposed on Fig. 4(a) to show the extent of the bowing of 
vertical lines. They are drawn so as to completely enclose 
one vertical black line.

The composite (sum) of these two images, shown in 
Fig. 4(c), reveals the extent of the distortion. There it can 
be seen that the distortion in Fig. 4(a) and (b) is apparently 
equal but in opposite directions, as illustrated in Fig. 2. The 
distortion is greatest at the edges, where the absolute difference 
in position $|x_2 - x_1|$ is approximately 5.8 mm. Dividing this 
motion by two we find that the positional error, $|x_2 - x_1| = 
x_2 - x_1$, in Fig. 4(a) and (b) is 2.9 mm near the edges. 
The error decreases to about half this size near the center. 
Along the vertical direction, however, the distortion decreases 
to zero as one approaches the edges. Fig. 4(c) provides a 
field map of sorts. Areas where the lines converge have 
small field inhomogeneity $|B_e|$; areas where the lines show 
large separation have a large $|B_e|$. We have compared this 
mapping qualitatively with Willcott’s direct non-imaging field 
mapping technique (not shown) [28] and have gotten excellent 
agreement [3].

Using our technique, we processed the two images of 
Fig. 4(a) and (b) to produce a rectified image. Based only on 
the knowledge that the readout gradients were horizontal and 
were equal in magnitude but reversed ($\alpha = -1$), the process 
yielded the image shown in Fig. 4(d). The overall distortions 
present in the unprocessed images has been greatly reduced 
or eliminated. The two white vertical lines in Fig. 4(d), cor-
responding to the similar lines in Fig. 4(a), show that the 
enclosed black line is now straight. At the resolution of these 
images, no remaining global geometrical distortion can be 
detected, but there are some noisy horizontal displacements 
that were not present in the unprocessed images. This noise is 
introduced by the numerical nature of our processing. These 
jagged displacements are in almost all cases less than or equal 
to one pixel.

The subject of our second experiment is a set of eight 
parallel cylindrical tubes of circular cross section immersed 
in tap water. Images were acquired in a plane perpendicular 
to the axes of the tubes and are shown in Fig. 5. The tubes contain
Fig. 4. Grid phantom geometrical distortion studies. (a) Image acquired with the readout gradient directed right to the left. (b) Image acquired with readout gradient directed left to right. (c) The superimposition of (a) and (b). (d) The rectified image.

either solutions of CuSO₄ or air and appear in the image plane in three groups—four in a central column, two to the left, and two to the right. We list the contents of the tubes in each group in top-to-bottom order: The tubes in the central column have 6 millimolar, air, 500 millimolar, and air. The two to the left have 125 millimolar and 250 millimolar. The two to the right have 1 molar and 750 millimolar. This phantom provides a convenient model for demonstrating intensity distortions caused by spatial fluctuations in $\beta$. Spatial variability is strongest near interfaces between media of differing susceptibilities. The combination of tubes and water bath provides eight such interfaces. The interiors of the tubes are chemically homogeneous, as is the water bath. Thus, any deviation from constant intensity within these regions indicates an intensity distortion. The imaging parameters are: spin-echo pulse sequence with $TE = 20$ ms, $TR = 500$ ms, readout gradient $= 1.57$ mT/m, slice selection gradient $= 4.8$ mT/m, and slice thickness $= 5$ mm. Pixel size was 1.17 mm. Fig. 5(a) and (b) show the unprocessed images, Fig. 5(a) with the readout gradient directed upward and Fig. 5(b) with the readout gradient directed downward. The phase encoding gradient was directed horizontally. The phase encoding and slice selection gradients were unchanged for both images. The three small highlights at the corners of the black arrowheads are intensity distortions. The deviation from circularity reveals geometrical distortion. The distortion into arrowhead shapes of circular cross sections of tubes containing air and CuSO₄ has been observed earlier by Lüdeke [15] and simulated by Posse [21]. Fig. 5(c) shows the result of subtracting Fig. 5(b) from Fig. 5(a). If there were no distortions, this image would be a uniform gray.

As with the grid phantom, we processed the two images of Fig. 5(a) and (b) to produce a rectified image. Based only on the knowledge that the readout gradients were vertical and were equal in magnitude but reversed ($\alpha = -1$), the process yielded the image shown in Fig. 5(d). The highlights present at the corners of the arrowheads are gone and the circular shapes of the tubes have been restored. There are some small light spots visible in the interiors of the tubes on the right in Fig. 5(d), which are presumably artifacts of the numerical processing. The extent of the intensity correction is not as obvious as that of the geometrical distortion. To measure it we computed some statistics on the intensity values within an annulus approximately two pixels wide around the lowest tube in Fig. 5(a), (b), and (d) just far enough from the tube itself to avoid partial volume effects. Table I contains the results. The coefficient of variation, $\sigma/\mu$, which would ideally be zero around the periphery of the tubes, serves as a measure of the
Fig. 5. Eight cylindrical tubes containing air or solutions of CuSO₄ as described in the text. The tubes are immersed in a bath of tap water. Images are acquired in a plane perpendicular to the tube axes. (a) Image acquired with readout gradient directed upward. (b) Image acquired with readout gradient downward. (c) The difference image, (a) minus (b). (d) The rectified image. The insets at the lower right of (a), (b), and (d) are enlargements of the bottom tube.

The approximation made in deriving the rectified images is in each case equivalent to ignoring the second order error term derived from (20) and (21). The results shown in these images suggest that the approximation is a good one for the gradients used in the image acquisition. For larger gradients the approximation would be even better.

### VI. SUMMARY AND CONCLUSION

We have presented a technique for accurate MR imaging in the presence of static field inhomogeneity. Such inhomogeneity produces both geometric distortion and intensity distortion. Our technique, which we call "rectification," produces an image free of both types of distortion. In this technique we acquire two distorted images using spin-echo pulse sequences that are identical except for a simple change in the gradients. We then combine the distorted images to obtain a single undistorted image. Rectification can be important in clinical applications.
imaging when extreme geometric accuracy is required for stereotactic procedures, in industrial nondestructive testing when a highly inhomogeneous field must be employed, and in microscopic imaging when high resolution is desired.

We have provided both a theoretical derivation and experimental evidence to show that the technique works. In our derivations we have confined our attention to a standard spin–echo pulse sequence and we have neglected the effects of T1 and T2 relaxation and of the finite slice profile. Our experimental images were each acquired with the same spin-echo sequences and reconstruction methods as well and are in fact negligible. Our technique will work also with other spin–echo sequences and reconstruction methods as well and might be modified to work with gradient refocusing. Work is in progress on this latter modification. Our experiments revealed that the technique occasionally introduces small artifacts into the low intensity regions of the image. We attribute that effect to the numerical nature of our image processing algorithm. The processing algorithms currently require the setting of a threshold to separate the region external to the imaged object from the outer surface of the object itself. Work is in progress on an altered algorithm that does not require this threshold.

We provided two versions of our rectification technique, an approximate technique for single slice images and an exact technique for volume images. We examined the extent of the approximation in the single slice method and found experimentally that the approximation is an excellent one.

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